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Constrained Systems: A Unified Geometric Approach

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A general geometric framework is devised in order to contain the presymplectic and Lagrangian formalisms as particular cases. We call these objects *constrained dynamical systems*, since their dynamics usually lead to *constraints*. Their most elementary properties are studied, and several related structures, especially morphisms, are defined. In particular, a stabilization algorithm is performed. As a byproduct, the dynamics and constraints of the Lagrangian formalism (with the "second-order condition") are intrinsically obtained.

1. INTRODUCTION

The presymplectic formalism is devised to geometrize the Hamiltonian formalism for constrained systems. For a presymplectic manifold (M, ω) the closed 2-form ω defines a morphism of vector M-bundles $\hat{\omega}: \mathrm{T}(M) \to \mathrm{T}(M)^*$. A "locally Hamiltonian constrained system" is then defined by a closed 1-form $\beta \in \Omega^1(M)$, and the equation of motion for a path ξ in M is

$$\hat{\omega} \circ \dot{\xi} = \beta \circ \xi$$

Since this equation is not written in normal form, a procedure to determine the subset of points in M by which solutions pass, and also the multiplicity of the solutions, is needed. This is achieved through a stabilization algorithm which is a geometrization of the Dirac-Bergmann constraint algorithm (Gotay *et al.*, 1978).

The presymplectic formalism has also been applied to the Lagrangian formalism, by adding the second-order condition to the Lagrangian presymplectic equation $\hat{\omega}_L \circ \dot{\xi} = dE_L \circ \xi$. Both equations can be equivalently written into a single one: there is a vector field K along the Legendre transformation

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FL [or, what is the same, a section of the inverse image of $\mathrm{T}(\mathrm{T}(Q)^*)$ under FL] such that the Euler–Lagrange equation for a path ξ in $\mathrm{T}(Q)$ can be written (Gràcia and Pons, 1989)

$$\mathrm{T}(\mathrm{F}L)\circ\dot{\xi} = K\circ\xi$$

When L is singular (i.e., FL is not a local diffeomorphism) a stabilization algorithm must be performed as in the presymplectic formalism.

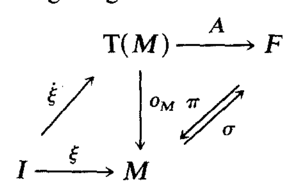
The dynamics above considered have a common feature: their equations of motion are not written in normal form, since $\dot{x}$ is multiplied by a linear operator depending on x. A geometric framework for such equations can be built as follows.

2. CONSTRAINED DYNAMICAL SYSTEMS

We consider a finite-dimensional paracompact manifold M, a vector bundle $\pi: F\to M$, a morphism of vector M-bundles $A: \mathrm{T}(M)\to F$, and a section $\sigma: M\to F$ of F. For a path $\xi: I\to M$ we consider the differential equation

$$A\circ\dot{\xi} = \sigma\circ\xi \tag{1}$$

In other words, the following diagram has to be commutative:



We shall call the quintuple (M, F, π, A, σ) a *constrained dynamical system*, and (1) the *equation of motion for a path* in M.

If the local expression of A is $(x, \mathbf{v})\mapsto(x, \mathbf{A}(x)\cdot\mathbf{v})$ and the local expression of σ is $x\mapsto(x, \boldsymbol{\sigma}(x))$, then the local expression of (1) is

$$\mathbf{A}(x(t))\cdot\dot{x}(t) = \boldsymbol{\sigma}(x(t))$$

In general, (1) may not have solutions passing through every point in M, and if there is a solution passing through x, it may not be unique. We call *motion set* the subset S of points $x\in M$ such that there is a solution ξ of the equation of motion passing by x. The ideal C of functions in M which vanish on the motion set S is called the *constraint ideal*, and its elements the *constraint functions*.

An alternative description of the dynamics can be given in terms of the *equation of motion for a vector field*: if $N\subset M$ is a submanifold contained in the motion set, and X is a vector field in M *tangent* to N, then the integral curves of X contained in N are solutions of the equation of motion (1) if and only if X satisfies

$$A \circ X \underset{N}{\simeq} \sigma \tag{2}$$

(where the notation $\simeq_N$ means equality at the points of N).

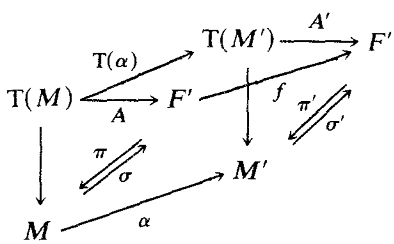
3. MORPHISMS OF CONSTRAINED DYNAMICAL SYSTEMS

A *morphism of constrained dynamical systems* between (M, F, π, A, σ) and $(M', F', \pi', A', \sigma')$ is just a morphism (α, f) between the vector bundles F and F' over M and M' such that

$$f \circ A = A' \circ \mathrm{T}(\alpha) \tag{3}$$

$$f \circ \sigma = \sigma' \circ \alpha \tag{4}$$

In other words, the following diagram has to be commutative:



With this definition of morphisms the constrained dynamical systems clearly constitute a category.

Many constructions with constrained dynamical systems can be carried on, as the inverse image through a mapping, the pass to the quotient in F, etc., and induce the corresponding morphisms. We point out only the following: if $j: M_0 \to M$ is a submanifold, we obtain a *subsystem* $(M_0, F|_{M_0}, \pi|_{M_0}, A|_{M_0}, \sigma|_{M_0})$ for which the natural inclusions yield a morphism into (M, F, π, A, σ).

A morphism (α, f) relates solutions and constraints in the following way:

1. If ξ is a path solution of the equation of motion in M, then $\xi' = \alpha \circ \xi$ is a solution in M'.
2. If ϕ' is a constraint in M', then $\phi = \alpha^*(\phi')$ is a constraint in M.

As a consequence, $\alpha(S) \subset S'$ and $\alpha^*(C') \subset C$.

4. THE STABILIZATION PROBLEM

If A is not an isomorphism, then the equation of motion (2) must be considered as an equation both for the submanifolds $N \subset M$ where the motion can take place and the vector fields X *tangent to* N.

In general, the motion set S is a union of submanifolds $N \subset M$, but sometimes S itself is a submanifold of M. Under some regularity conditions it can be determined as follows.

Let us assume that $\operatorname{Ker} A \subseteq \mathrm{T}(M)$ and $\operatorname{Im} A \subseteq F$ are *vector subbundles.* This amounts to saying that A has locally constant rank.

The equation $A_x \cdot X(x) = \sigma(x)$ for the unknown vector $X(x)$ can be solved only at the points $x \in M$ such that the compatibility condition

$$\sigma(x) \in \operatorname{Im} A_x \tag{5}$$

holds. This is equivalent to saying that $\sigma(x)$ is orthogonal to the kernel of the transposed morphism,

$$\langle \operatorname{Ker} {}^t A_x, \sigma(x) \rangle = 0 \tag{6}$$

Let $M_1 = \{x \in M \mid \sigma(x) \in \operatorname{Im} A_x\}$. It is a closed subset of M, since $\operatorname{Im} A \subset F$ is closed. If $(s^a)_{1 \leq a \leq m}$ is a local frame for $\operatorname{Ker} {}^t A$, then $M_1 \subset M$ is locally described, using (6), by the vanishing of the *m primary constraint functions*

$$\phi^a := \langle s^a, \sigma \rangle \tag{7}$$

As usual in the theory of constrained systems, we shall assume that M_1 is a submanifold, called the *primary constraint submanifold.* Similar assumptions will be made in what follows.

The compatibility condition (5) has led to the consideration of solutions X which satisfy (2) only on M_1; therefore, we are interested only in the values of $X_1 = X|_{M_1}$. Since X (or X_1) must be *tangent to* M_1, the initial problem becomes *the same problem for the subsystem defined by* M_1:

$$\begin{array}{ccc} \mathrm{T}(M_1) & \xrightarrow{A_1} & F_1 \\ \downarrow & \pi_1 \swarrow\!\!\nearrow \sigma_1 & \\ M_1 & & \end{array}$$

where $F_1 = F|_{M_1}$, and π_1, A_1, and σ_1 are the corresponding restrictions to M_1.

Now we repeat the procedure. The compatibility condition for this system yields a subset M_2, also assumed to be a submanifold. In general, let us write $M_0 = M$, $A_i = A|_{T(M_i)}$, and $\sigma_i = \sigma|_{M_i}$, considered as mappings into $F_i = F|_{M_i}$, and define recursively

$$M_{i+1} := \{x \in M_i \mid \sigma_i(x) \in \operatorname{Im} A_{ix}\} \tag{8}$$

(all these submanifolds can be also described in terms of constraints, using frames for Ker ${}^tA_i$). This sequence ends on the *final constraint submanifold* M_f. The result is

$$\begin{array}{ccccc} \mathrm{T}(M_f) & \xrightarrow{A_f} & \mathrm{Im}\, A_f & \longrightarrow & F_f \\ \big\downarrow & \nearrow_{\sigma_f} & & & \\ M_f & & & & \end{array}$$

and therefore the equation $A_f \circ X_f = \sigma_f$ for a vector field X_f in M_f has solutions; these are not unique when Ker $A_f \neq 0$.

This stabilization algorithm can be given a form more suitable for computations. One can also show that morphisms preserve each step of the stabilization algorithms and therefore the corresponding constraints.

When our constrained system corresponds to the presymplectic formalism, then the algorithm we have presented is equivalent to that of Gotay *et al.* (1978). When the Lagrangian formalism is considered as in the introduction (Gràcia and Pons, 1989), we obtain an intrinsic stabilization algorithm which yields the constraints and the dynamics. Earlier work on the topic either were coordinate dependent and relied on the Hamiltonian stabilization algorithm (Batlle *et al.*, 1986), relied on the presymplectic Lagrangian formalism (Muñoz and Román-Roy, 1989), or, moreover, included a gauge fixing (Gotay and Nester, 1980). As an example, the primary Lagrangian constraints are directly obtained from (7): they are $\chi_\mu := K \cdot \phi_\mu$, where ϕ_μ run over the constraints defined by $FL(\mathrm{T}Q)$ in T^*Q.

5. COMMENTS AND APPLICATIONS

The theory of presymplectic manifolds is not general enough to include directly the Lagrangian formalism deduced from a singular Lagrangian—the "second-order differential equation" condition must be explicitly imposed. Therefore, we have introduced a natural extension of both presymplectic and Lagrangian formalisms, under the name of *constrained dynamical systems*, and the subsequent concepts of morphisms and constraints; other constrained systems also covered by our formalism are quoted below. We have carefully related the equations of motion for paths and vector fields, a point which is sometimes missed in the literature.

In order to solve the equation of motion, a "stabilization algorithm" is devised. It has the advantage of being recursive: each step yields a new subsystem where a similar equation of motion has to be solved. For presymplectic systems it is equivalent to that of Gotay *et al.* (1978), but now its main interest is to yield an intrinsic stabilization algorithm for the Lagrangian formalism (including the "second-order condition").

The morphisms allow us to relate different constrained systems and their equations of motion, constraints, and stabilization algorithms. In particular, the equivalence between Lagrangian and Hamiltonian formalisms through the Legendre transformation (Batlle *et al.*, 1986) can be geometrically formulated. On the other hand, the Dirac Hamiltonian formalism (Dirac, 1964) can be regarded as a constrained dynamical system, which is easily shown to be *isomorphic* to the presymplectic Hamiltonian formalism (Gotay *et al.*, 1978).

As another application of our generalized framework, the equations of motion (and the constraints in the singular case) for higher-order Lagrangians have been studied (Gràcia *et al.*, 1989), as well as the different "mth-order differential equation" conditions that can be considered on them (Gràcia *et al.*, 1989).

These results, together with other related questions, additional examples, and detailed proofs, will be given elsewhere (Gràcia and Pons, 1991).

We hope that our formalism will be useful in a deeper study of constrained systems, especially the Lagrangian formalism and symmetries.

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